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Discrete semi-classical orthogonal polynomials: generalized Charlier

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Abstract

Generalized Charlier polynomials are introduced as semi-classical orthogonal polynomials of class 1 with one parameter. Main characteristic data are established from the Laguerre–Freud equations generating the coefficients of the recurrence relation satisfied by the polynomials.

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1. r-generalized Charlier polynomials

Classical continuous and discrete orthogonal polynomials are defined by a positive weight $\rho \equiv \rho(x)$, solution of a Pearson differential or difference equation given, respectively, by

$$D(\sigma(x)\rho(x)) = \tau(x)\rho(x), \quad \Delta(\sigma(x)\rho(x)) = \tau(x)\rho(x), \tag{1.1}$$

where $Df = \mathrm{d}f/\mathrm{d}x = f'$, $\Delta f(x) = f(x+1) - f(x)$; $\sigma(x)$ is a polynomial of degree $\leqslant 2$ and $\tau(x)$ of degree 1. A striking difference between continuous and discrete case arises from the following considerations:

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Let us consider in both cases the equation satisfied by a new weight

$$\bar{\rho} \equiv \bar{\rho}(x) = [\rho(x)]^{\lambda}, \quad \lambda > 0.$$

In the continuous case, using differential equation (1.1)

$$\frac{\bar{\rho}'}{\bar{\rho}} = \lambda \frac{\rho'}{\rho} = \lambda \frac{\tau - \sigma'}{\sigma} = \frac{\lambda\tau - (\lambda - 1)\sigma' - \sigma'}{\sigma} \tag{1.2}$$

and the power of the classical weight is still a classical weight which satisfies:

$$(\overline{\sigma\rho})' = \overline{\tau\rho} \quad \text{with } \bar{\sigma} = \sigma \quad \text{and} \quad \bar{\tau} = \lambda\tau - (\lambda - 1)\sigma' \tag{1.3}$$

($\bar{\tau}$ staying of first degree).

The discrete case is completly different. From (1.1)

$$\frac{\rho(x+1)}{\rho(x)} = \frac{\sigma(x) + \tau(x)}{\sigma(x+1)} \tag{1.4}$$

and now the λ power of this discrete weight becomes solution of

$$\frac{\bar{\rho}(x+1)}{\bar{\rho}(x)} = \left[\frac{\sigma(x) + \tau(x)}{\sigma(x+1)}\right]^{\lambda} \tag{1.5}$$

which is never classical, and only semi-classical if λ is a positive integer. Classical Charlier polynomials correspond to $\sigma(x)=x$ and $\tau(x)=\mu-x$ with its orthogonal support being all integer i, ($\mu > 0$). The solution of (1.4) is given by Nikiforov and Uvarov [4]

$$\rho(i) = \frac{\mu^i}{i!} \quad i = 0, 1, \dots \tag{1.6}$$

The r-generalized Charlier polynomials are defined from Eq. (1.1) with

$$\sigma(x) = x^r, \ \tau(x) = \mu - x^r, \ r \text{ is a positive integer,} \tag{1.7}$$

giving now for the weight ρ_r

$$\frac{\rho_r(x+1)}{\rho_r(x)} = \frac{\mu}{(x+1)^r}, \tag{1.8}$$

the solution being

$$\rho_r(i) = \frac{\mu^i}{(i!)^r}, \quad i = 0, 1, \dots\ . \tag{1.9}$$

For $r = 2$, this weight is semi-classical of class $s = 1$ and the moments $M_k(\mu) = \sum_{i=0}^{\infty} i^k \mu^i/(i!)^2$ can be expressed in terms of modified Bessel functions I_k, $k = 0$ and 1

$$M_0(\mu) = I_0(2\sqrt{\mu}), \quad M_1(\mu) = \sqrt{\mu} I_1(2\sqrt{\mu}). \tag{1.10}$$

The recurrence relation for the moments easily obtained from Eq. (1.9) reads

$$M_{k+2}(\mu) = \mu M_k(\mu) - \sum_{i=1}^{k} \binom{k}{i} (-1)^i M_{k+2-i}(\mu). \tag{1.11}$$

The corresponding power of a Charlier weight (of parameter $\sqrt{\mu}$) is given by

$$\rho_2^{(\mu)}(i) = [\rho^{(\sqrt{\mu})}(i)]^2, \quad i = 0, 1, \dots\ . \tag{1.12}$$

These generalized Charlier polynomials with $r = 2$ denoted $\bar{C}_n^{(\mu)}(x)$ cannot be obtained from any weight modification of the Charlier polynomials $C_n^{\mu}(x)$, in contrast to the generalized Meixner case introduced in [5].

2. Recurrence coefficients for the generalized Charlier orthogonal polynomials ($r = 2$)

Let $(P_n)_{n\geqslant 0}$ be a family of monic polynomials of degree exactly equal to n and orthogonal with respect to a positive weight $\rho(x)$. The three-term recurrence relation satisfied by the family $(P_n)_{n\geqslant 0}$ writes as

$$\begin{aligned} &P_{n+1}(x) = (x-\beta_n)P_n(x) - \gamma_n P_{n-1}(x), \quad && n\geqslant 1 \\ &P_0(x) = 1,\ P_1(x) = x-\beta_0,\ \gamma_n \neq 0 && \text{for all } n\geqslant 1. \end{aligned} \tag{2.1}$$

The family of orthogonal polynomials $(P_n)_{n\geqslant 0}$ is said semi-classical of class s, with respect to a difference operator Δ [2,3], if

$$\Delta(\sigma\rho) = \tau\rho, \tag{2.2}$$

$\tau(x)$ and $\sigma(x)$ being given polynomials of fixed degree with

$$s = \max\{[\text{degree of } \tau(x)] - 1,\ [\text{degree of } \sigma(x)] - 2\}.$$

A crucial problem consists to deduce the recurrence coefficients β_n, γ_n as function of the coefficients contained in $\tau(x)$ and $\sigma(x)$ [1–3]. This leads to a coupled algebraic system of equations giving β_n and γ_n that are called Laguerre–Freud equations [2,3].

For

$$\tau(x) = \mu - x^2, \quad \sigma(x) = x^2, \tag{2.3}$$

the Laguerre–Freud equations for the generalized Charlier polynomials ($r = 2$) reduce to

$$\gamma_{n+1} = \sum_{i=0}^{n-1}\beta_i + n\beta_n - \gamma_n - \beta_n^2 - \binom{n}{2} + \mu, \quad n\geqslant 1,\ (\gamma_0 \equiv 0) \tag{2.4}$$

$$(\beta_{n+1}+\beta_n)\gamma_{n+1} = \sum_{i=0}^{n}\beta_i^2 + 2\sum_{k=1}^{n}\gamma_k - n\sum_{i=0}^{n}\beta_i + (n+1)\gamma_{n+1} + \binom{n+1}{3}, \quad n > 1, \tag{2.5}$$

with the following initial values:

$$\beta_0 = \frac{M_1}{M_0} = \sqrt{\mu}\,\frac{I_1(2\sqrt{\mu})}{I_0(2\sqrt{\mu})}; \quad \gamma_1 = \mu - [\beta_0]^2, \tag{2.6}$$

M_0 and M_1 being the moments of zero- and first-order, respectively. Then the coefficient γ_{n+1} can be uniquely expressed as function of β_n such that

$$\begin{aligned} \gamma_{n+1} = {}& n\sum_{i=0}^{n}(-1)^i\beta_{n-i} + 2\sum_{i=1}^{N} i(\beta_{n+1-2i} - \beta_{n-2i}) \\ & - \sum_{i=0}^{n}(-1)^i\beta_{n-i}^2 - \sum_{i=0}^{n-2}(-1)^i\binom{n-i}{2} + \mu - \beta_0^2 \end{aligned} \tag{2.7}$$

with $N = \text{integer}[n/2, (n+1)/2]$. The computation of β_n and γ_n values versus n for various μ, using computer algebra, leads to the following remarks:

(i) For each given μ, the function γ_n versus n asymptotically tends to μ, and this asymptotic value is reached as shorter as μ is smaller; in particular for $\mu \leqslant 0.1$, $\gamma_n = \mu = \text{constant}\ \forall n$;

(ii) For each given μ, the function β_n versus n evolves to become equal to n and this equality occurs as more rapidly as smaller μ; in particular, for smaller μ (typically for $\mu \leqslant 0.1$), this function equals to n ($\beta_n \equiv n$) $\forall n$.

Whence, the three-term recurrence relation (2.1) satisfied by the family of the 2-generalized monic Charlier semi-classical orthogonal polynomials should be written as

$$P_{n+1}(x) = (x-n)P_n(x) - \mu P_{n-1}(x), \tag{2.8}$$

when $n \to \infty$.

Let mention that β_n and γ_n can be explicitly expressed in a representation involving exclusively the relevant parameters μ and M_1 and M_0. For example, the first coefficients, write as

$$\begin{aligned}
\gamma_1 &= \mu - \beta_0^2,\\
\beta_1 &= \frac{\mu - \beta_0 + \beta_0^3}{\mu - \beta_0^2},\\
\gamma_2 &= \frac{-\mu\beta_0(2\beta_0^2 + \beta_0 - 2\mu)}{(\beta_0^2 - \mu)^2},\\
\beta_2 &= \frac{3\beta_0^5 + (2\mu+1)\beta_0^4 - 3\mu\beta_0^3 - 4\mu^2\beta_0^2 + 2\mu^3}{\beta_0(\beta_0^2 - \mu)(2\beta_0^2 + \beta_0 - 2\mu)}.
\end{aligned} \tag{2.9}$$

3. Structure relations and second-order difference equation

As any semi-classical orthogonal family, $P_n(x)$ satisfies a structure relation which for the class $s = 1$ reduces to

$$\sigma(x)\nabla P_n(x) = \sum_{i=n-2}^{n+2} \lambda_{n,i} P_i(x), \tag{3.1}$$

where $\lambda_{n,i}$ are constants. These five coefficients can be computed [6] in general from the recurrence coefficients γ_n, β_n, and in the generalized Charlier case ($r = 2$) reduce to ($\lambda_{n,n+2} \equiv 0$)

$$\lambda_{n,n+1} = n;\ \ \lambda_{n,n} = -\binom{n}{2} + n\beta_n + \sum_{i=0}^{n-1}\beta_i;\ \ \lambda_{n,n-1} = \gamma_n(\beta_{n-1} + \beta_n);\ \ \lambda_{n,n-2} = \gamma_{n-1}\gamma_n. \tag{3.2}$$

Using basic recurrence (2.1), structure relation (3.1) can be reduced to

$$\sigma(x)\nabla P_n(x) = N_n(x)P_{n-1}(x) + M_n(x)P_n(x), \tag{3.3}$$

or, equivalently, to a similar equation using the following relation between ∇ and Δ:

$$\nabla P_n(x) = \Delta P_n(x) - \Delta\nabla(P_n(x)) = \Delta P_n(x-1), \tag{3.4}$$

where

$$M_n(x) = \lambda_{n,n} - \frac{\lambda_{n,n-2}}{\gamma_{n-1}} + (x - \beta_n)\lambda_{n,n+1}$$
$$N_n(x) = \lambda_{n,n-1} + \frac{\lambda_{n,n-2}}{\gamma_{n-1}}(x - \beta_{n-1}) - \lambda_{n,n+1}\gamma_n. \tag{3.5}$$

In the same way, one can use another structure relation given by

$$(\sigma(x) + \tau(x))\Delta P_n(x) = \sum_{i=n-2}^{n-1} \tilde{\lambda}_{n,i} P_i(x) \tag{3.6}$$

such that

$$\alpha(x) = \sigma(x) + \tau(x) = \mu \tag{3.7}$$

and

$$\tilde{\lambda}_{n,n-1} = n\mu, \tag{3.8}$$

$$\tilde{\lambda}_{n,n-2} = \mu \left[\binom{n}{2} - n\beta_{n-1} + \sum_{i=0}^{n-1} \beta_i \right]. \tag{3.9}$$

The structure relation (3.6) thus rewrites as

$$\mu\Delta P_n(x) = \tilde{N}_n(x)P_{n-1}(x) + \tilde{M}_n(x)P_n(x), \tag{3.10}$$

with

$$\tilde{M}_n(x) = -\frac{\tilde{\lambda}_{n,n-2}}{\gamma_{n-1}}$$

$$\tilde{N}_n(x) = \tilde{\lambda}_{n,n-1} + \frac{\tilde{\lambda}_{n,n-2}}{\gamma_{n-1}}(x - \beta_{n-1}). \tag{3.11}$$

Using now Eqs. (3.3), (3.4) and (3.10), and after some trivial algebra consisting to eliminate $P_{n-1}(x)$ from the equations, we obtain the following second-order difference equation:

$$\left\{ \left[\sum_{i=0}^{n-2} \beta_i + (1-n)\beta_{n-1} + \binom{n}{2} \right] x^3 + \left[n\gamma_{n-1} + (n-1)\beta_{n-1}^2 \right.\right.$$
$$\left.\left. -\beta_{n-1} \left(\sum_{i=0}^{n-2} \beta_i + \binom{n}{2} \right) \right] x^2 \right\} \Delta\nabla P_n(x)$$
$$+ \left\{ \left[(n-1)\beta_{n-1} - \sum_{i=0}^{n-2} \beta_i - \binom{n}{2} \right] x^3 + \left[\beta_{n+1} \left(\sum_{i=0}^{n-2} \beta_i + \binom{n}{2} \right) \right.\right.$$
$$\left.\left. +(1-n)\beta_{n-1}^2 - n\gamma_{n-1} \right] x^2 + \gamma_{n-1}\gamma_n x + \gamma_{n-1}\gamma_n(\beta_n - n) \right\} \Delta P_n(x)$$

$$
\begin{aligned}
&+\Bigg\{\left[n\sum_{i=0}^{n-2}\beta_i+(n-n^2)\beta_{n-1}+n\binom{n}{2}\right]x^2\\
&+\left[\sum_{i=0}^{n-2}\beta_i^2+2\sum_{0\leqslant i<j\leqslant n-1}\beta_i\beta_j+(n-1)^2\beta_{n-1}^2-2n\beta_{n-1}\sum_{i=0}^{n-2}\beta_i+n^2\gamma_{n-1}-\binom{n}{2}^2\right]x\\
&+\gamma_n\beta_n\left[\sum_{i=0}^{n-2}\beta_i+(1-n)\beta_{n-1}+\binom{n}{2}\right]+\gamma_n\left[\left(n^2+\binom{n}{2}\right)\beta_{n-1}\right.\\
&\left.-n\left(\beta_{n-1}^2+\gamma_{n-1}+\binom{n}{2}\right)\right]+\beta_{n-1}^2\left[n\left(\sum_{i=0}^{n-1}\beta_i-\binom{n}{2}\right)\right]\\
&+\beta_{n-1}\left[\gamma_n\sum_{i=0}^{n-1}\beta_i-\left(\sum_{i=0}^{n-1}\beta_i\right)^2+\binom{n}{2}^2\right]-n\binom{n}{2}\gamma_{n-1}\Bigg\}P_n(x)=0
\end{aligned}
$$

satisfied by the 2-generalized Charlier polynomials.

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References

[1] S. Belmehdi, Formes Linéaires et Polynômes Orthogonaux Semi-Classiques de classe $s = 1$. Description et Classification, Thèse d'Etat, Université P. et M. Curie, Paris VI, 1990.

[2] S. Belmehdi, A. Ronveaux, Laguerre–Freud's equations for the recurrence coefficients of semi-classical orthogonal polynomials, J. Approx. Theory 76 (1994) 351–368.

[3] M. Foupouagnigni, M.N. Hounkonnou, A. Ronveaux, Laguerre–Freud equations for the recurrence coefficients of D_ω-semi-classical orthogonal polynomials of class one, J. Comput. App. Math. 99 (1998) 143–154.

[4] A.F. Nikiforov, V.B. Uvarov, Special Functions of Mathematical Physics, Birkhäuser, Basel, 1988.

[5] A. Ronveaux, Discrete semi-classical orthogonal polynomials: generalized Meixner, J. Approx. Theory 46 (4) (1986) 403–407.

[6] L. Salto Díaz, Polinomios D_ω-Semiclásicos, Tesis Doctoral, Universidad de Alcalá de Henares, 1995.